

Birth-and-death Pólya urns and stationary random partitions

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We introduce a class of birth-and-death Pólya urns, which allow for both sampling and removal of observations governed by an auxiliary inhomogeneous Bernoulli process, and investigate the asymptotic behaviour of the induced allelic partitions. By exploiting some embedded models, we show that the asymptotic regimes exhibit a phase transition from partitions with almost surely infinitely many blocks and independent counts, to stationary partitions with a random number of blocks. The first regime corresponds to limits of Ewens-type partitions and includes a result of [Arratia, Barbour and Tavaré \(1992\)](#) as a special case. We identify the invariant and reversible measure in the second regime, which preserves asymptotically the dependence between counts, and is shown to be a mixture of Ewens sampling formulas, with a tilted Negative Binomial mixing distribution on the sample size.

Keywords: Ewens sampling formula, phase transition, allelic partition, mixture model, particle process, sampling.

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1 Introduction and outline of the results

Pólya urn schemes provide easily interpretable exchangeable sequences and are among the most celebrated sampling rules in probability. See [Johnson and Kotz \(1977\)](#) and [Mahmoud \(2009\)](#) for general treatments. Of particular interest for our purposes is the Blackwell–MacQueen Pólya urn ([Blackwell and MacQueen, 1973](#)): given $\lambda > 0$ and a nonatomic probability measure P_0 on a Polish space \mathbb{X} , a sequence sampled from a Pólya urn is such that $X_1 \sim P_0$ and for $n \geq 1$

$$(1) \quad X_{n+1} \mid X_1, \dots, X_n \sim \frac{\lambda}{\lambda + n} P_0 + \frac{1}{\lambda + n} \sum_{i=1}^n \delta_{X_i},$$

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where δ_x denotes a point mass at x . Since P_0 has no atoms, if X_{n+1} is sampled from P_0 a new value is observed, otherwise X_{n+1} is a copy of a previous observation. Hence a Pólya urn sample will feature ties with positive probability, inducing a partition of the observed values. A popular interpretation of the above scheme is as a species sampling model (Pitman, 1996), whereby the observations label species sampled from a large population, and those drawn from P_0 are species that have not been previously observed.

The impact of the Blackwell–MacQueen Pólya urn schemes and its developments has been extremely significant in applied probability and statistics, particularly through the construction and characterisation of random probability measures via limits of exchangeable sequences (Blackwell and MacQueen, 1973; Pitman, 1995; 1996; 2006; Gneden and Pitman, 2005; Lijoi, Mena and Prünster, 2005; 2007), and as a building block in the architecture of computational strategies for posterior inference with Bayesian nonparametric mixture models (Escobar and West, 1995; MacEachern and Müller, 1998; Neal, 2000; Ishwaran and James, 2001).

In these respects, particularly relevant is its relationship with the Ewens sampling formula, which assigns probability

$$(2) \quad \text{ESF}_n(m_1, \dots, m_n) = \frac{n! \Gamma(\lambda)}{\Gamma(\lambda + n)} \prod_{i=1}^n \left(\frac{\lambda}{i} \right)^{m_i} \frac{1}{m_i!} \mathbb{1} \left\{ \sum_{j=1}^n j m_j = n \right\},$$

to vectors $m = (m_1, \dots, m_n) \in \mathbb{Z}_+^n$, where $\mathbb{1}(\cdot)$ is the indicator function. Originally introduced for describing the sampling distribution of allelic frequencies in a neutral population at equilibrium (Ewens, 1972), this provides the law of the “allelic” partition (m_1, \dots, m_n) induced by a Pólya urn sample of size n , where m_i is the number of alleles appearing exactly i times. Equivalently, it provides the law of the partition induced by sampling from a Dirichlet process random probability measure (Antoniak, 1974). See Crane (2016) (with discussion) for a recent review of applications and connections of the Ewens sampling formula.

In this paper we consider a class of birth-and-death Pólya urns (B&D-PUs for short), which in addition to adding observations according to (1) allow to remove observations from the current sample, and investigate their asymptotic regimes under a certain specification of the probability of a removal step. Rather than extending the predictive distribution in (1), we define these directly in terms of the dynamics induced on the associated allelic partition. For any $\beta \in (0, 1]$, define a B&D-PU as the Markov chain $M = \{M(h), h \in \mathbb{Z}_+\}$ with state space \mathbb{Z}_+^∞ and transition probabilities

$$(3) \quad p(m'|m) \propto \begin{cases} \beta \lambda, & m' = m + e_1, \\ \beta i m_i, & m' = m - e_i + e_{i+1}, \quad i \geq 1 \\ (1 - \beta) i m_i, & m' = m - e_i + e_{i-1}, \quad i \geq 1 \end{cases}$$

where \propto denotes proportionality and

$$m \pm e_i = (m_1, \dots, m_{i-1}, m_i \pm 1, m_{i+1}, \dots), \quad m \pm e_0 = m.$$

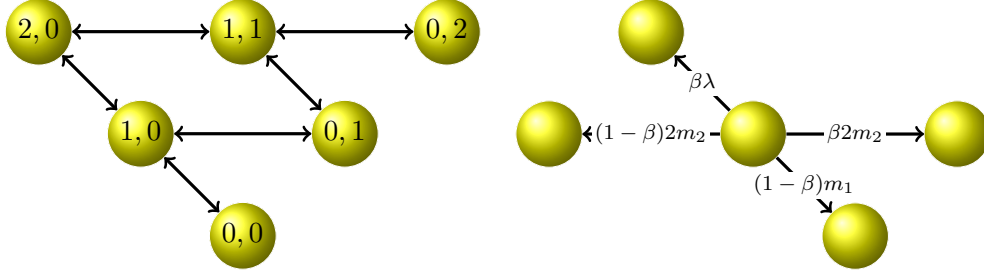


Figure 1: Graph representation of the partition-valued process induced by B&D-PUs restricted to (m_1, m_2) (left), and probabilities of admissible transitions, up to proportionality (right).

Note that the normalising constant in (3) is $\beta\lambda + \sum_{i \geq 1} im_i$. Without loss of generality, we let for convenience M and other auxiliary chains introduced later start from the origin $(0, 0, \dots)$, instead of assigning them an initial distribution. The transitions in (3) correspond respectively to the introduction of a new block of size one; to a block of size i becoming of size $i + 1$; and to a block of size i becoming of size $i - 1$. Here im_i is the total number of items in blocks of size i . The dynamics of M , restricted to its first two coordinates, are depicted on a lattice in Figure 1. Equivalently, the above transitions can be expressed in terms of the underlying process of observations, whereby with probability

$$(4) \quad b(\beta, m) = \frac{\beta(\lambda + \sum_{i \geq 1} im_i)}{\beta\lambda + \sum_{i \geq 1} im_i}$$

a further observation is drawn from (1) and added to the sample, and with probability $1 - b(\beta, m)$ an observation is chosen uniformly from the current sample and removed. Setting $\beta = 1$ above reduces (3) to the usual dynamics induced on partitions by the Pólya urn (1) (cf., e.g., Feng (2010), Section 2.7.2), whereas $\beta = 0$ would simply remove sequentially all items currently available until none is left, hence it is not considered here. Note that the random allelic partitions induced by Pólya urns are consistent under uniform deletion, i.e., the partition obtained by removing a uniformly chosen item from an ESF_n -distributed partition of n elements has distribution ESF_{n-1} ; cf., e.g., Crane (2016). Hence perturbing the Pólya urn dynamics by a finite number of uniform removals does not effect its limiting behaviour. Here, however, we are allowing for an infinite number of removals according to an auxiliary inhomogeneous Bernoulli process with state-dependent probability $1 - b(\beta, m)$, and study the implied long run behaviour.

We show that B&D-PUs exhibit asymptotically a phase transition at $\beta = 1/2$ from stationary dynamics with a random number of blocks $k = \sum_i m_i$, to nonstationary dynamics which yield almost surely infinitely-many blocks. To this aim, we first study the stationary properties of

an auxiliary system of finite-dimensional Markov chains on the space of partitions. These differ from other commonly used partition-valued processes, e.g., those in [Petrov \(2009\)](#), which are indexed by a fixed sample size, or those in [Crane \(2014\)](#), which are indexed by the maximum number of blocks. Our auxiliary processes are instead indexed by a maximal allelic count, i.e., the maximum number of items allowed in each block, whereas both the number of blocks and the sample size are left free to vary. The ergodicity of these chains is then exploited by embedding them, at certain stopping times, in B&D-PUs, whose asymptotic distributions coincide with the weak limits of the stationary laws of the auxiliary chains. Specifically, values of $\beta \in [1/2, 1]$ for B&D-PUs in the long run generate infinite structures analogous to those induced by limits of Ewens partitions. In this limit, the allelic counts (m_1, m_2, \dots) are asymptotically independent with Poisson distribution of mean λ/i , irrespective of the value of β . This result for $\beta = 1$ was first proved by [Arratia, Barbour and Tavaré \(1992\)](#). Values of $\beta \in (0, 1/2)$ are instead shown to generate stationary models on \mathbb{Z}_+^∞ with invariant and reversible distribution

$$(5) \quad \pi(m) = \frac{\beta\lambda + \sum_{i \geq 1} im_i}{\beta\lambda + \beta\lambda/(1-2\beta)} \prod_{i \geq 1} \text{Po}(m_i; \theta_i/i), \quad \theta_i = \lambda \left(\frac{\beta}{1-\beta} \right)^i,$$

where $\text{Po}(\cdot; \theta)$ is a Poisson probability mass function with mean θ . Here several elements are of interest. The first is the dependence between the allelic counts m_i , which, contrary to the $\beta \geq 1/2$ case, is preserved in the limit. An interpretation for this dependence can be given by taking m_i be independent Poisson variables of mean θ_i/i , with θ_i as in (5), and letting $J = j$ with probability proportional to θ_j for $j \geq 1$ or to $\beta\lambda$ for $j = 0$. Then the vector $(m_1, \dots, m_{j-1}, m_j + 1, m_{j+1}, \dots)$ has distribution (5). The second is the fact that the expected number of items $\mathbb{E}(im_i)$ in groups of size i can be easily checked to be proportional to θ_i and thus depends on i , whereas in the Ewens case $\mathbb{E}(im_i) = \lambda$ in the limit. Recently, [Betz and Ueltschi \(2011\)](#); [Betz, Ueltschi and Velenik \(2011\)](#) studied the asymptotic behaviour of generalised sampling formulas where the counts distribution also depends on the count index i , obtained by replacing λ in (2) with λ_i for a sequence $(\lambda_1, \lambda_2, \dots)$ of nonnegative reals. A third element of interest is the fact that the number of groups at stationarity is random and finite, with distribution determined by (β, λ) ; see (20) below. [Gnedin \(2010\)](#) studied a partition structure generated sequentially which yields a finite, random number of blocks. Finally, in this stationary regime, the underlying system of particles is also stationary with invariant measure related to a mixture of Pólya urn schemes, with a tilted Negative Binomial mixing distribution on the sample size. The latter is defined here as the total number of items in the system at a given time, i.e., $n := \sum_{i \geq 1} im_i$.

The paper is organised as follows. Section 2 defines the auxiliary system of Markov chains with maximal allelic counts and identifies their invariant measure. Section 3 proves the phase transition for the partition structures generated asymptotically by B&D-PUs, identifies their limiting distributions and shows the reversibility for $\beta \in (0, 1/2)$. Finally, Section 4 highlights

a connection of our results with a mixture of Ewens sampling formulas.

2 Chains with maximal allelic count

In this Section we define and study a system of finite-dimensional partition-valued Markov chains, which are instrumental for the investigation of the B&D-PUs asymptotic regimes. Fix $L \in \mathbb{N}$ and $\mu > 0$, and define $M_L = \{M_L(h), h \in \mathbb{Z}_+\}$ to be the \mathbb{Z}_+^L -valued Markov chain with transition probabilities

$$(6) \quad p_L(m'|m) \propto \begin{cases} \beta\lambda, & m' = m + e_1, \\ \beta im_i, & m' = m - e_i + e_{i+1}, \quad i = 1, \dots, L-1, \\ \beta Lm_L, & m' = m - e_L, \\ (1-\beta)im_i, & m' = m - e_i + e_{i-1}, \quad i = 1, \dots, L, \\ (1-\beta)\mu, & m' = m + e_L. \end{cases}$$

with normalising constant $\beta\lambda + (1-\beta)\mu + |m|$, for $|m| = \sum_{i \geq 1} im_i$. Here the difference with respect to (3) is that the system has a maximal allelic count m_L , whereby a group of size L which becomes of size $L+1$ is removed from the system, with probability proportional to βLm_L , and groups of size L can be inserted in the system, with probability proportional to $(1-\beta)\mu$.

The following result identifies the invariant measure of M_L .

Theorem 2.1. *M_L with transition probabilities (6) has unique invariant distribution*

$$(7) \quad \pi_L(m) = \frac{\beta\lambda + (1-\beta)\mu + |m|}{\beta\lambda + (1-\beta)\mu + \sum_{i=1}^L \theta_i} \prod_{i=1}^L \text{Po}(m_i; \theta_i/i),$$

where

$$(8) \quad \theta_i = w_i(\beta)\lambda + (1 - w_i(\beta))\mu, \quad w_i(\beta) = \begin{cases} \beta^i \frac{(1-\beta)^{L-i+1} - \beta^{L-i+1}}{(1-\beta)^{L+1} - \beta^{L+1}}, & \beta \neq 1/2, \\ \frac{L-i+1}{L+1}, & \beta = 1/2. \end{cases}$$

Proof. Let

$$(9) \quad \tilde{\pi}_L(m) = \prod_{i=1}^L \text{Po}(m_i; \theta_i/i).$$

The global balance condition reads

$$\sum_{m \in \mathbb{Z}_+^L} (\beta\lambda + (1-\beta)\mu + |m|) \tilde{\pi}_L(m) p_L(m'|m)$$

$$\begin{aligned}
&= \beta \left(\tilde{\pi}_L(m' - e_1)\lambda + \sum_{i=1}^{L-1} \tilde{\pi}_L(m' + e_i - e_{i+1})i(m'_i + 1) + \tilde{\pi}_L(m' + e_L)L(m'_L + 1) \right) \\
&\quad + (1 - \beta) \left(\tilde{\pi}_L(m' + e_1)(m'_1 + 1) + \sum_{i=2}^L \tilde{\pi}_L(m' + e_i - e_{i-1})i(m'_i + 1) + \tilde{\pi}_L(m' - e_L)\mu \right) \\
(10) \quad &= \tilde{\pi}_L(m') \left[\beta \left(\frac{\lambda}{\theta_1} m'_1 + \sum_{i=1}^{L-1} \frac{\theta_i}{i(m'_i + 1)} \frac{(i+1)m'_{i+1}}{\theta_{i+1}} i(m'_i + 1) + \frac{\theta_L}{L(m'_L + 1)} L(m'_L + 1) \right) \right. \\
&\quad \left. + (1 - \beta) \left(\frac{\theta_1}{m'_1 + 1} (m'_1 + 1) + \sum_{i=2}^L \frac{\theta_i}{i(m'_i + 1)} \frac{(i-1)m'_{i-1}}{\theta_{i-1}} i(m'_i + 1) + \frac{Lm'_L}{\theta_L} \mu \right) \right] \\
&= \tilde{\pi}_L(m') \left[\beta \theta_L + (1 - \beta) \theta_1 + m'_1 \left(\frac{\beta \lambda + (1 - \beta) \theta_2}{\theta_1} \right) \right. \\
&\quad \left. + \sum_{i=2}^{L-1} i m'_i \left(\frac{\beta \theta_{i-1} + (1 - \beta) \theta_{i+1}}{\theta_i} \right) + L m'_L \left(\frac{\beta \theta_{L-1} + (1 - \beta) \mu}{\theta_L} \right) \right].
\end{aligned}$$

The right hand side equals $\tilde{\pi}_L(m')(\beta \lambda + (1 - \beta) \mu + |m'|)$ upon imposing

$$\begin{cases} \beta \lambda + (1 - \beta) \theta_2 = \theta_1, \\ \beta \theta_{i-1} + (1 - \beta) \theta_{i+1} = \theta_i, \quad i = 2, \dots, L-1, \\ \beta \theta_{L-1} + (1 - \beta) \mu = \theta_L, \\ \beta \theta_L + (1 - \beta) \theta_1 = \beta \lambda + (1 - \beta) \mu. \end{cases}$$

The last equation equals the sum of the first L , hence $(\theta_1, \dots, \theta_L)$ is the solution of the system of L linear equations

$$\begin{cases} \theta_1 - (1 - \beta) \theta_2 = \beta \lambda \\ \theta_i - \beta \theta_{i-1} - (1 - \beta) \theta_{i+1} = 0, \quad i = 2, \dots, L-1, \\ \theta_L - \beta \theta_{L-1} = (1 - \beta) \mu. \end{cases}$$

Lemma A.1 in the Appendix now implies that the solution of the system is given by (8), and the statement follows by dividing both sides of (10) by the normalising constant

$$\sum_{m \in \mathbb{Z}_+^L} (\beta \lambda + (1 - \beta) \mu + |m|) \tilde{\pi}_L(m) = \beta \lambda + (1 - \beta) \mu + \sum_{i=1}^L \theta_i.$$

Finally, uniqueness follows from positive recurrence, which can be easily proved. \square

An interpretation for the dependence among the counts m_i at stationarity can be provided by means of an alternative representation for the invariant distribution. Let m_i be mutually

independent each with $\text{Po}(m_i; \theta_i/i)$ distribution, θ_i as in (8), and select $J = j$ with probability proportional to θ_j for $j = 1, \dots, L$ and to $\beta\lambda + (1-\beta)\mu$ for $j = 0$. Then $m' = (m_1, \dots, m_{j-1}, m_j + 1, m_{j+1}, \dots, m_L)$ has distribution (7). This can be easily shown by exploiting the fact that if Z has $\text{Po}(z; \lambda)$ distribution, then $Z + 1$ has probability mass function $\text{Po}(z; \lambda)z/\lambda$. An informal interpretation for the component-specific parameters θ_i 's in (8) can be provided by recalling that new groups enter the system from the left (i.e., $m \mapsto m + e_1$) with probability proportional to λ and from the right (i.e., $m \mapsto m + e_L$) with probability proportional to μ . It is easily checked that $w_i(\beta)$ is decreasing in i . Then (8) expresses the fact that the effect of λ (resp. μ) on m_i is stronger for small (resp. large) i . For odd L , the median parameter $\theta_{(L+1)/2}$ simplifies to

$$(11) \quad w_{(L+1)/2}(\beta) = \frac{\beta^{(L+1)/2}}{\beta^{(L+1)/2} + (1-\beta)^{(L+1)/2}}, \quad \beta \neq 1/2,$$

which shows more explicitly the effect of β on the median count and its dependence on the number of counts separating it from the extremal m_1 and m_L (see the proof of Lemma A.1). When $\beta = 1/2$, (11) further simplifies to $1/2$, and $\theta_{(L+1)/2}$ reduces to $(\lambda + \mu)/2$.

3 Birth-and-death Pólya urns

B&D-PUs have been defined in the Introduction to be partition-valued Markov chains M with state space \mathbb{Z}_+^∞ and transition probabilities (3). Informally, the underlying sampling process can be thought of as Pólya urn sampling where particles are deleted at random times. Here we exploit the class of chains with maximal allelic count M_L , introduced in the previous section, for identifying the asymptotic regimes of M . The strategy is to let a sequence of chains with maximal capacity $\{M_L, L \in \mathbb{N}\}$ converge to the B&D-PU M as $L \rightarrow \infty$, and then obtain the asymptotic regimes as appropriate limits of the marginal distributions of M_L . We achieve this by letting the probability of introducing L -sized blocks in M_L be governed by μ_L instead of μ (cf. (6)), and letting μ_L converge to zero appropriately fast as $L \rightarrow \infty$. The key intuition here is that since the probability of blocks entering the system for left and right is proportional to $\beta\lambda$ and $(1-\beta)\mu_L$ respectively, the expected number of items entering the system from left and right is proportional to $\beta\lambda$ and $(1-\beta)L\mu_L$ respectively, so the second term needs to go to zero appropriately fast, as $L \rightarrow \infty$, in order to obtain asymptotically well-defined dynamics.

First we identify, with the following result, the weak limits of the invariant distribution of M_L , determined in Theorem 2.1, as μ_L goes to 0 with L .

Theorem 3.1. *Let $\{\mu_L\}_{L \geq 1} \subset \mathbb{R}_+$ be decreasing and such that $L\mu_L \rightarrow 0$ as $L \rightarrow \infty$, and let $(Z_1^{(L)}, \dots, Z_L^{(L)})$ have distribution (7), with θ_i as in (8) replacing μ with μ_L . Then, as $L \rightarrow \infty$,*

$$(Z_1^{(L)}, Z_2^{(L)}, \dots) \xrightarrow{d} (Z_1, Z_2, \dots),$$

where (Z_1, Z_2, \dots) :

(i) are independent Poisson random variables with mean λ/i , if $\beta \in [1/2, 1]$;

(ii) have joint distribution (5), if $\beta \in (0, 1/2)$.

Proof. Let $w_i(\beta)$ be as in (8). As $L \rightarrow \infty$, it is easy to see that, for any i , $w_i(\beta) \rightarrow (\beta/(1-\beta))^i$ when $\beta < 1/2$ and $w_i(\beta) \rightarrow 1$ when $\beta \geq 1/2$. Also, let $\theta_i^{(L)}$ be as in (8) with $\mu = \mu_L$. Since $\mu_L \rightarrow 0$ as $L \rightarrow \infty$, $\theta_i^{(L)} \rightarrow (\beta/(1-\beta))^i \lambda$ when $\beta < 1/2$ and $\theta_i^{(L)} \rightarrow \lambda$ when $\beta \geq 1/2$. Then, by using (27) of Lemma A.1 in the Appendix, as $L \rightarrow \infty$ we have

$$\sum_{i=1}^L \theta_i^{(L)} = \begin{cases} L\mu_L + \frac{\beta}{1-2\beta}(\lambda - \mu_L) + o(L\mu_L), & \text{if } \beta < 1/2, \\ \frac{\lambda + \mu_L}{2}L, & \text{if } \beta = 1/2, \\ \lambda L + \frac{1-\beta}{1-2\beta}(\lambda - \mu_L) + o(\mu_L), & \text{if } \beta > 1/2. \end{cases}$$

Thus $L\mu_L \rightarrow 0$ implies that $\sum_{i=1}^L \theta_i^{(L)}$ converges to $\lambda\beta/(1-2\beta)$ when $\beta < 1/2$ since, and diverges for $\beta \geq 1/2$.

Denote now by \mathbb{E}_L , $\tilde{\mathbb{E}}_L$ and $\tilde{\mathbb{E}}_\infty$ the expectations with respect to π_L in (7) and $\tilde{\pi}_L, \tilde{\pi}_\infty$ in (9), respectively, with parameters $\theta_i^{(L)}$ in π_L and $\tilde{\pi}_L$, and $\theta_i = (\beta/(1-\beta))^i \lambda$ for $\beta < 1/2$ and $\theta_i = \lambda$ for $\beta \geq 1/2$ in $\tilde{\pi}_\infty$. For any sequence $\{\phi_i\}$ such that

$$\begin{cases} \sum_{i \geq 1} \phi_i \left(\frac{\beta}{1-\beta} \right)^i \frac{1}{i} < \infty, & \beta < 1/2 \\ \sum_{i \geq 1} \frac{\phi_i}{i} < \infty, & \beta \geq 1/2, \end{cases}$$

we have

$$\begin{aligned} \mathbb{E}_L \left(e^{-\sum_{i \geq 1} \phi_i m_i} \right) &= \frac{\beta\lambda + (1-\beta)\mu_L}{\beta\lambda + (1-\beta)\mu_L + \sum_{i=1}^L \theta_i^{(L)}} \tilde{\mathbb{E}}_L \left(e^{-\sum_{i=1}^L \phi_i m_i} \right) \\ &\quad + \frac{1}{\beta\lambda + (1-\beta)\mu_L + \sum_{i=1}^L \theta_i^{(L)}} \sum_{i=1}^L \tilde{\mathbb{E}}_L \left(e^{-\sum_{i=1}^L \phi_i m_i} i m_i \right) \\ (12) \quad &= \frac{\beta\lambda + (1-\beta)\mu_L}{\beta\lambda + (1-\beta)\mu_L + \sum_{i=1}^L \theta_i^{(L)}} \tilde{\mathbb{E}}_L \left(e^{-\sum_{i=1}^L \phi_i m_i} \right) \\ &\quad + \frac{1}{\beta\lambda + (1-\beta)\mu_L + \sum_{i=1}^L \theta_i^{(L)}} \sum_{i=1}^L \theta_i^{(L)} e^{-\phi_i} \tilde{\mathbb{E}}_L \left(e^{-\sum_{i=1}^L \phi_i m_i} \right) \\ &= \frac{\beta\lambda + (1-\beta)\mu_L + \sum_{i=1}^L \theta_i^{(L)} e^{-\phi_i}}{\beta\lambda + (1-\beta)\mu_L + \sum_{i=1}^L \theta_i^{(L)}} \tilde{\mathbb{E}}_L \left(e^{-\sum_{i=1}^L \phi_i m_i} \right) \end{aligned}$$

where the second and third equalities follow from

$$\begin{aligned}\mathbb{E}(e^{-\phi X}) &= \exp\{-\theta(1 - e^{-\phi})\} \\ \mathbb{E}(Xe^{-\phi X}) &= -\frac{d}{d\phi}\mathbb{E}(e^{-\phi X}) = \theta e^{-\phi} \exp\{-\theta(1 - e^{-\phi})\},\end{aligned}$$

for $X \sim \text{Po}(x; \theta)$. Here

$$\lim_{L \rightarrow \infty} \tilde{\mathbb{E}}_L \left(e^{-\sum_{i=1}^L \phi_i m_i} \right) = \tilde{\mathbb{E}}_\infty \left(e^{-\sum_{i \geq 1} \phi_i m_i} \right)$$

as long as

$$\lim_{L \rightarrow \infty} \sum_{i=1}^L \frac{\theta_i^{(L)}}{i} (1 - e^{-\phi_i}) = \sum_{i \geq 1} \frac{\theta_i}{i} (1 - e^{-\phi_i}).$$

The latter is in turn implied by $\theta_i^{(L)} \rightarrow \theta_i$ and an application of the monotone convergence theorem, since $\theta_i^{(L-1)} \leq \theta_i^{(L)}$ for L large when μ_L decreases to 0; see Lemma A.2 in the Appendix. As for the first factor on the right hand side of (12), an application of Cesaro's Theorem, together with the fact that $\beta \geq 1/2$, $\sum_{i=1}^L \theta_i^{(L)} \rightarrow \infty$ and $\phi_i \rightarrow 0$, yields

$$\lim_{L \rightarrow \infty} \frac{\beta\lambda + (1 - \beta)\mu_L + \sum_{i=1}^L \theta_i^{(L)} e^{-\phi_i}}{\beta\lambda + (1 - \beta)\mu_L + \sum_{i=1}^L \theta_i^{(L)}} = 1.$$

When $\beta < 1/2$ and $\lim_{L \rightarrow \infty} L\mu_L = 0$,

$$(13) \quad \lim_{L \rightarrow \infty} \frac{\beta\lambda + (1 - \beta)\mu_L + \sum_{i=1}^L \theta_i^{(L)} e^{-\phi_i}}{\beta\lambda + (1 - \beta)\mu_L + \sum_{i=1}^L \theta_i^{(L)}} = \frac{\beta\lambda + \sum_{i \geq 1} \theta_i e^{-\phi_i}}{\beta\lambda + \beta\lambda/(1 - 2\beta)},$$

where at the numerator we have applied the monotone convergence theorem. Noting that

$$\frac{\beta\lambda + \sum_{i \geq 1} \theta_i e^{-\phi_i}}{\beta\lambda + \beta\lambda/(1 - 2\beta)} \tilde{\mathbb{E}}_\infty \left(e^{-\sum_{i \geq 1} \phi_i m_i} \right)$$

corresponds to the Laplace transform of m under the distribution (5) completes the proof. \square

Note that, when $\beta \geq 1/2$, the weaker assumption that $\mu_L \rightarrow 0$ suffices for the above result. This is informally due to the fact that $\beta \geq 1/2$ makes the addition of size-1 blocks to the system frequent enough to counterbalance the frequency of L -sized blocks entering the system from the right when $\mu_L \rightarrow 0$, instead of $L\mu_L \rightarrow 0$. When $\beta < 1/2$, this is not the case and μ_L must go to zero faster than $1/L$.

Next, we exploit an embedding of M_L in M at appropriate stopping times, in order to show that the weak limits in Theorem 3.1 describe the long-time behaviour of the B&D-PU. Note that, when $\beta \geq 1/2$, the result extends Theorem 1 in [Arratia, Barbour and Tavaré \(1992\)](#).

Theorem 3.2. *Let M be a B&D-PU with transitions (3), and let $Z_j(h)$ its j -th component. Then, as $h \rightarrow \infty$,*

$$(Z_1(h), Z_2(h), \dots) \xrightarrow{d} (Z_1, Z_2, \dots)$$

where (Z_1, Z_2, \dots) :

- (i) *have joint distribution (5), for $\beta \in (0, 1/2)$;*
- (ii) *are independent Poisson random variables with mean λ/i , for $\beta \in [1/2, 1]$.*

Proof. Let $\rho_L : \mathbb{Z}_+^\infty \rightarrow \mathbb{Z}_+^L$, defined as

$$(14) \quad \rho_L(m) = (m_1, \dots, m_L),$$

be the restriction of $m \in \mathbb{Z}_+^\infty$ to its first L components. We show that, for any $N \in \mathbb{N}$, $\rho_N(M(h)) \xrightarrow{d} (Z_1, \dots, Z_N)$ as $h \rightarrow \infty$ in the two regimes. To this end, define the auxiliary chains $\tilde{M}_L = \{\tilde{M}_L(h), h \in \mathbb{Z}_+\}$ on \mathbb{Z}_+^∞ , with transition probabilities

$$(15) \quad \tilde{p}_L(m'|m) \propto \begin{cases} \beta\lambda, & m' = m + e_1, \\ \beta i m_i, & m' = m - e_i + e_{i+1}, \quad i \geq 1, \\ (1 - \beta) i m_i, & m' = m - e_i + e_{i-1}, \quad i \neq L + 1, \\ (1 - \beta) \mu_L, & m' = m - e_{L+1} + e_L. \end{cases}$$

and normalising constant $\beta\lambda + (1 - \beta)\mu_L + |m| - (1 - \beta)(L + 1)m_{L+1}$. Here \tilde{M}_L differs from a B&D-PU in that transitions $m \mapsto m - e_{L+1} + e_L$, whereby one item from an $(L + 1)$ -sized group is removed, have probability proportional to $(1 - \beta)\mu_L$ instead of $(1 - \beta)(L + 1)m_{L+1}$. Note that count m_L and m_{L+1} remain dependent, in view of the transition $m \mapsto m - e_L + e_{L+1}$. Let $\{\mu_L\}_{L \geq 1}$ be a decreasing sequence such that, as $L \rightarrow \infty$, $L\mu_L \rightarrow 0$. Let

$$(16) \quad \sigma_k = \min \{h > \sigma_{k-1} : \rho_L(\tilde{M}_L(h)) \neq \rho_L(\tilde{M}_L(\sigma_{k-1}))\},$$

to be the k th time a transition of \tilde{M}_L involves the first L counts. Proposition A.3 in the Appendix shows that M_L is embedded in \tilde{M}_L at the stopping times σ_k , in that $\{\sigma_k, n \geq 1\}$ occur infinitely often and

$$\mathbb{P} \left(\rho_L(\tilde{M}_L(\sigma_k)) = m' \mid \rho_L(\tilde{M}_L(\sigma_{k-1})) = m \right) = p_L(m'|m),$$

with p_L as in (6) with $\mu = \mu_L$. Together with Theorem 2.1, this implies

$$\rho_L \left(\tilde{M}_L(\sigma_k) \right) \xrightarrow{d} (Z_1^{(L)}, \dots, Z_L^{(L)}), \quad \text{as } k \rightarrow \infty,$$

where the right hand side has distribution π_L with $\mu = \mu_L$, see (7). Clearly, this also implies

$$(17) \quad \rho_N \left(\tilde{M}_L(\sigma_k) \right) \xrightarrow{d} (Z_1^{(L)}, \dots, Z_N^{(L)}), \quad \text{as } n \rightarrow \infty$$

for any $N \leq L$. Emphasising the dependence on L in (16) by $\sigma_k^{(L)}$, note now that $\{\sigma_k^{(L)}, n \geq 1\} \subset \{\sigma_n^{(L+1)}, n \geq 1\}$ and $\{\sigma_k^{(L)}, n \geq 1\} \uparrow \mathbb{N}$ as $L \rightarrow \infty$ with probability one, since for all $h \in \mathbb{N}$ there exists an L_0 such that $\{1, \dots, h\} \subset \{\sigma_k^{(L)}, n \geq 1\}$ for all $L \geq L_0$. Therefore, for any given h ,

$$\rho_N \left(\tilde{M}_L(h) \right) = \rho_N \left(\tilde{M}_L(\sigma_h) \right)$$

for L sufficiently large. The result now follows by taking the limit for $L \rightarrow \infty$ on both sides of (17), in virtue of Theorem 3.1. \square

The asymptotic regimes of B&D-PUs are thus determined by the probability of introducing new singleton blocks into the system. These produce for $\beta \geq 1/2$ infinite partitions analogous to those induced by Pólya urns, since insertion of singletons are frequent enough to make deletions asymptotically irrelevant, and the counts m_i become independent in the limit. When $\beta < 1/2$, instead, the stream of incoming items is not frequent enough and the dependence between the counts m_i is retained in the limit with distribution (5). The next result shows that this latter case provides the reversible and invariant distribution of B&D-PU with $\beta \in (0, 1/2)$.

Theorem 3.3. *Let M have transitions as in (3) with $\beta \in (0, 1/2)$. Then (5) is the reversible and invariant measure of M .*

Proof. Let $\tilde{\pi}_\infty(m)$ be as in (9) and $C^{-1} = \beta\lambda + \beta\lambda/(1 - 2\beta)$ be the normalizing constant appearing in (5). Then, for any $m \in \mathbb{Z}_+^\infty$, we have

$$\begin{aligned} \pi(m + e_1) p(m \mid m + e_1) &= C \tilde{\pi}_\infty(m + e_1) (1 - \beta) (m_1 + 1) \\ &= C \tilde{\pi}_\infty(m) \frac{\theta_1}{m_1 + 1} (1 - \beta) (m_1 + 1) = C \tilde{\pi}_\infty(m) \beta \lambda = \pi(m) p(m + e_1 \mid m) \end{aligned}$$

and for any $i \geq 1$

$$\begin{aligned} \pi(m - e_i + e_{i+1}) p(m \mid m - e_i + e_{i+1}) &= C \tilde{\pi}_\infty(m - e_i + e_{i+1}) (1 - \beta) (i + 1) (m_{i+1} + 1) \\ &= C \tilde{\pi}_\infty(m) \frac{i m_i}{\theta_i} \frac{\theta_{i+1}}{(i + 1) (m_{i+1} + 1)} (1 - \beta) (i + 1) (m_{i+1} + 1) \\ &= C \tilde{\pi}_\infty(m) \beta i m_i = \pi(m) p(m - e_i + e_{i+1} \mid m) \end{aligned}$$

yielding the result. Finally, in view of the positive recurrence of the chain, which can be easily proved, $\pi(m)$ is also the unique invariant measure of M . \square

With a similar argument to that used in Section 2 for the chains with maximal allelic count, the invariant distribution of B&D-PU's admits representation as augmented vector of independent Poisson variables. Recalling that $\theta_i = (\beta/(1-\beta))^i \lambda$, define $J = j$ with probability proportional to θ_j for $j \geq 1$ and proportional to $\beta\lambda$ for $j = 0$. One can easily check, using the geometric series $\sum_{i \geq 1} p^i = p/(1-p)$ for $0 < p < 1$, that

$$(18) \quad \mathbb{P}(J = 0) = \frac{1}{2} \left(1 - \frac{\beta}{1-\beta} \right), \quad \mathbb{P}(J = j) = \frac{1}{2\beta} \left(1 - \frac{\beta}{1-\beta} \right) \left(\frac{\beta}{1-\beta} \right)^j, \quad j \geq 1.$$

This can be obtained from a geometric distribution of parameter $1 - \beta/(1-\beta)$, by reallocating half of the mass assigned to 0 to the other support points, resulting in a tilting given by the factor $1/2\beta$. Next, if $m = (m_1, m_2, \dots)$ where the m_i 's are independent each with $\text{Po}(m_i; \theta_i/i)$ distribution, then

$$(19) \quad m + e_J \sim \pi(m)$$

for $\pi(m)$ in (5). Note also that in the stationary regime, the random partition induced by B&D-PU's has a random number of groups $K := \sum_i m_i$, contrary to the number of groups induced by usual Pólya urns, recovered here for $\beta \geq 1/2$, which grows to infinity asymptotically as $\log h$ (Korwar and Hollander, 1973). Exploiting the representation (18)–(19) and using the fact that $\sum_{i \geq 1} p^i/i = -\log(1-p)$ for $0 < p < 1$, one finds that, when $\beta < 1/2$,

$$(20) \quad K \stackrel{d}{=} K_\infty + \mathbb{1}(J \geq 1), \quad K_\infty \sim \text{Pois}(k; -\lambda \log(1 - \beta/(1-\beta)))$$

where K_∞ corresponds to the sum of independent Poisson random variables with parameters θ_i as in the l.h.s. of (19). This immediately yields the moments of K , for example

$$\mathbb{E}(K) = \mathbb{P}(J = 0) - \lambda \log(1 - \beta/(1-\beta)).$$

4 Connection with a mixture of Ewens sampling formulas

We conclude by showing that the invariant measure of B&D-PU's corresponds to a mixture of Ewens sampling formulas ESF_n in (2) with a tilted Negative Binomial mixing measure on the sample size n . Let

$$(21) \quad \text{NB}(n; r, p) = \frac{\Gamma(r+n)}{n! \Gamma(r)} p^n (1-p)^r, \quad n = 0, 1, \dots$$

be the Negative Binomial distribution with parameters $r > 0$ and $p \in (0, 1)$.

Theorem 4.1. *Let $\beta \in (0, 1/2)$ and M be a B&D-PU with transition probabilities (3) and invariant distribution π as in (5). Then*

(i) $N := \sum_{i \geq 1} im_i$ is a birth-and-death chain with invariant distribution

$$\mu(n) \propto (\beta\lambda + n) \text{NB}(n; \lambda, \beta/(1 - \beta)), \quad n = 0, 1, \dots;$$

(ii) π admits representation as mixture of Ewens sampling formulas

$$(22) \quad \pi(m) = \sum_{n \geq 0} \text{ESF}_n(\rho_n(m)) \mu(n), \quad m \in \mathbb{Z}_+^\infty,$$

with ρ_n as in (14).

Note that (22) is well defined for $n = 0$ if one interprets (2) as giving probability one to the empty vector $\rho_0(m) = \emptyset$, as one can easily verify that $\pi(0, 0, \dots) = \mu(0)$.

Proof. From (3), it is easily seen that N is a birth-and-death chain with immigration, whose transitions $n \rightarrow n \pm 1$ have probabilities $t(n \pm 1|n)$ proportional to $\beta(\lambda + n)$ and $(1 - \beta)n$ respectively. The expected increment of N is proportional to $\beta\lambda - (1 - 2\beta)n$, which is always positive for $\beta \geq 1/2$, yielding non stationarity. For $\beta < 1/2$, using the fact that the expected value of the Negative Binomial distribution in (21) is $pr/(1 - p)$, it is readily verified that

$$(23) \quad \begin{aligned} \mu(n) &= \frac{\beta\lambda + n}{2\beta\lambda/(1 - \beta/(1 - \beta))} \text{NB}(n; \lambda, \beta/(1 - \beta)) \\ &= \frac{\beta\lambda + n}{2\beta\lambda} \frac{\lambda_{(n)}}{n!} \left(\frac{\beta}{1 - \beta} \right)^n \left(1 - \frac{\beta}{1 - \beta} \right)^{\lambda+1} \end{aligned}$$

where $a_{(x)} = a(a + 1) \cdots (a + x - 1) = \Gamma(a + x)/\Gamma(a)$ is the Pochhammer symbol. The detailed balance condition for N then reads

$$(24) \quad \begin{aligned} \mu(n - 1) t(n|n - 1) &= \frac{\beta\lambda + n - 1}{2\beta\lambda} \frac{\lambda_{(n-1)}}{(n - 1)!} \left(\frac{\beta}{1 - \beta} \right)^{n-1} \left(1 - \frac{\beta}{1 - \beta} \right)^{\lambda+1} \frac{\beta(\lambda + n - 1)}{\beta\lambda + n - 1} \\ &= \frac{\beta\lambda + n}{2\beta\lambda} \frac{\lambda_{(n)}}{n!} \left(\frac{\beta}{1 - \beta} \right)^n \left(1 - \frac{\beta}{1 - \beta} \right)^{\lambda+1} \frac{(1 - \beta)n}{\beta\lambda + n} = \mu(n) t(n - 1|n), \end{aligned}$$

hence μ is the reversible measure for N , yielding the first assertion. To prove (22), it suffices to show that $\pi(m) = \text{ESF}_n(\rho_n(m)) \mu(n)$ whenever $m \in \mathbb{Z}_+^\infty$ is such that $\sum_{i \geq 1} im_i = n$, for ESF_n and $\mu(n)$ as in (2) and (23). Let $p = \beta/(1 - \beta)$ and $C^{-1} = \beta\lambda + \beta\lambda/(1 - 2\beta) = 2\beta\lambda/(1 - \beta/(1 - \beta))$ be the normalising constant appearing both in (5) and in the first display of (23). Assuming

that $\sum_{i \geq 1} im_i = n$, we have

$$\begin{aligned} \pi(m) &= C \left(\beta\lambda + \sum_{i=1}^n im_i \right) \prod_{i \geq 1} \frac{1}{m_i!} \left(\frac{\lambda p^i}{i} \right)^{m_i} e^{-\lambda p^i/i} \\ &= C(\beta\lambda + n) p^n \exp \left\{ -\lambda \sum_{i \geq 1} \frac{p^i}{i} \right\} \prod_{i=1}^n \left(\frac{\lambda}{i} \right)^{m_i} \frac{1}{m_i!} \\ &= C(\beta\lambda + n) \left(\frac{\beta}{1-\beta} \right)^n \left(1 - \frac{\beta}{1-\beta} \right)^\lambda \prod_{i=1}^n \left(\frac{\lambda}{i} \right)^{m_i} \frac{1}{m_i!} \end{aligned}$$

where in the third equality we have used the fact that $\sum_{i \geq 1} p^i/i = -\log(1-p)$. Multiplying and dividing by $\lambda_{(n)}/n!$ now gives the desired condition. \square

For what concerns the sampling process associated to B&D-PUs, say X , which alternates sampling of observations from (1) to removals of uniformly chosen observations, this evolves in $E = \emptyset \cup (\cup_{n \geq 1} \mathbb{X}^n)$ according to the following transition probabilities

$$(25) \quad q(x'|x) \propto \begin{cases} \beta\lambda, & \text{if } x' = (x_1, \dots, x_N, y), \quad y \sim P_0, \\ \beta, & \text{if } x' = (x_1, \dots, x_j, \dots, x_N, x_j), \quad 1 \leq j \leq N, \\ (1-\beta), & \text{if } x' = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_N), \quad 1 \leq j \leq N, \end{cases}$$

with normalising constant $\beta\lambda + N$. It is immediate from Theorem 4.1 to see that this particle process is also stationary when $\beta < 1/2$ with invariant measure given by the mixture of Pólya urn schemes

$$\text{PU}(\text{d}x) := \sum_{n \geq 0} \text{PU}_n(\text{d}x) \mu(n),$$

where μ is as in Theorem 4.1, $\text{PU}_n(\text{d}x)$ represents the joint law of X_1, \dots, X_N drawn from the Pólya urn scheme (1), conditional on $N = n$, and PU_0 assigns probability one to the empty set.

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Appendix

Lemma A.1. *Let $\lambda, \mu > 0$. Then the system*

$$\begin{cases} \theta_1 - (1 - \beta)\theta_2 = \beta\lambda \\ \theta_i - \beta\theta_{i-1} - (1 - \beta)\theta_{i+1} = 0, & i = 2, \dots, L-1, \\ \theta_L - \beta\theta_{L-1} = (1 - \beta)\mu. \end{cases}$$

has solution

$$(26) \quad \theta_i = \begin{cases} \beta^i \frac{(1 - \beta)^{L-i+1} - \beta^{L-i+1}}{(1 - \beta)^{L+1} - \beta^{L+1}} \lambda + (1 - \beta)^{L-i+1} \frac{(1 - \beta)^i - \beta^i}{(1 - \beta)^{L+1} - \beta^{L+1}} \mu, & \beta \neq 1/2, \\ \frac{L - i + 1}{L + 1} \lambda + \frac{i}{L + 1} \mu, & \beta = 1/2. \end{cases}$$

Moreover

$$(27) \quad \sum_{i=1}^L \theta_i = \begin{cases} \frac{(1 - \beta)^{L+1} \mu - \beta^{L+1} \lambda}{(1 - \beta)^{L+1} - \beta^{L+1}} L + \frac{(1 - \beta)\beta}{1 - 2\beta} \frac{(1 - \beta)^L - \beta^L}{(1 - \beta)^{L+1} - \beta^{L+1}} (\lambda - \mu), & \beta \neq 1/2, \\ \frac{\lambda + \mu}{2} L, & \beta = 1/2. \end{cases}$$

Proof. The system can be written $A\mathbf{x} = \boldsymbol{\beta}$, where $\boldsymbol{\beta} = \beta\lambda\mathbf{e}_1 + (1 - \beta)\mu\mathbf{e}_L$ and A is a tridiagonal matrix with entries $-(1 - \beta), 1, -\beta$ respectively above, on and below the main diagonal. Let $d_L = \det(A)$ where subscript L refers to the dimension of A . Then d_l , $1 \leq l \leq L$, corresponds to the determinant of the $l \times l$ submatrix made of the first l rows and columns of A , i.e. $\det(A)$ when $L = l$. By using expansion by the first column, it is easy to check that d_l satisfies the recursive relation

$$(28) \quad d_l = d_{l-1} - \beta(1 - \beta)d_{l-2}, \quad l \geq 2$$

with initial values $d_1 = 1, d_2 = 1 - \beta(1 - \beta)$. Consider the associated second order difference equation

$$(29) \quad x_{t+2} - x_{t+1} + \beta(1 - \beta)x_t, \quad t \geq 2$$

i.e. (28) with $d_t = x_{t-1}$. It has characteristic equation $m^2 - m + \beta(1 - \beta) = 0$ with roots $m_1 = b$ and $m_2 = (1 - \beta)$. When $\beta \neq 1/2$, (29) has general solution given by

$$x_t = c_1\beta^t + c_2(1 - \beta)^t, \quad c_1, c_2 \in \mathbb{R}$$

so that by using the initial conditions $x_0 = 0$ and $x_1 = 1 - \beta(1 - \beta)$, we find $c_2 = (1 - \beta)^2/(1 - 2\beta)$, $c_1 = 1 - c_2 = -\beta^2/(1 - 2\beta)$ and, in turn,

$$(30) \quad d_l = ((1 - \beta)^{l+1} - \beta^{l+1})/(1 - 2\beta)$$

When $\beta = 1/2$, (29) has general solution given by $x_t = (c_1 + c_2 t)(1/2)^t$, $c_1, c_2 \in \mathbb{R}$ where $1/2$ is the common root of the characteristic equation. By using the initial conditions $x_0 = 0$ and $x_1 = 1 - \beta(1 - \beta) = 1 - 1/4$, we find $c_1 = 1$; $c_2 = 1/2$, therefore

$$(31) \quad d_l = \left(1 + \frac{l-1}{2}\right) 2^{-(l-1)} = (l+1)2^{-l}.$$

Note that $d_l \neq 0$ for any l since the constant solution of (29), $x_t = 0$, is ruled out by the initial condition $x_0 = 1$. Moreover, $d_l > \beta(1 - \beta)d_{l-1}$, so that $d_l > 0$ for any l , cfr. (28). In particular, $d_l > \beta^{l-1}(1 - \beta)^{l-1}$, $l \geq 2$. Since for $\beta \neq 0, 1$ the absolute value of the roots of the characteristic equation is < 1 , the constant solution $x_t = 0$ is stable, meaning that $x_t \rightarrow 0$ as $t \rightarrow \infty$, i.e. $\lim_{l \rightarrow \infty} d_l = 0$. When $\beta = 0, 1$, (29) has constant solution $x_t = 1$, hence $d_l = 1$ for any l .

Since now $d_L \neq 0$, the solution is unique and given by

$$\theta = A^{-1}\beta = \beta A^{-1}e_1\lambda + (1 - \beta)A^{-1}e_L\mu$$

and, in particular,

$$(32) \quad \theta_i = \beta a_{i1}\lambda + (1 - \beta)a_{iL}\mu, \quad i = 1, \dots, L$$

where a_{ij} is the (ij) th entry of A^{-1} . By using Cramer's method,

$$A^{-1} = \frac{1}{\det A} ((-1)^{i+j} \det A_{ij})^T$$

where A_{ij} is the $(L-1) \times (L-1)$ matrix obtained by deletion of the i th row and j th column of A . Hence,

$$a_{i1} = d_L^{-1}(-1)^{1+i} \det A_{1i}, \quad a_{iL} = d_L^{-1}(-1)^{L+i} \det A_{Li}$$

Consider a_{iL} first. It is easy to see that A_{L1} is lower triangular with $-(1 - \beta)$ in the diagonal, hence $\det(A_{L1}) = [-(1 - \beta)]^{L-1}$. For $1 < i < L$, A_{Li} is a $(L-1) \times (L-1)$ matrix that can be partitioned as

$$A_{Li} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

where B_{11} correspond to A with $i-1$ rows, B_{12} is a $(i-1) \times (L-i)$ made of all zero and B_{22} is a lower triangular matrix with $L-i$ rows and $-(1 - \beta)$ in the diagonal. In particular,

$\det(B_{11}) = d_{i-1}$ and $\det(B_{22}) = [-(1-\beta)]^{L-i}$. By using the formula for the determinant of partitioned matrices,

$$\det \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \det(B_{11}) \det(B_{22} - B_{21}B_{11}^{-1}B_{12})$$

we find that

$$\det A_{Li} = \det(B_{11}) \det(B_{22}) = d_{i-1}(-1)^{L-i}(1-\beta)^{L-i}$$

so that

$$a_{iL} = d_L^{-1}(-1)^{L+i}d_{i-1}(-1)^{L-i}(1-\beta)^{L-i} = d_L^{-1}(-1)^{2L}d_{i-1}(1-\beta)^{L-i} = d_L^{-1}d_{i-1}(1-\beta)^{L-i}$$

Finally, A_{LL} corresponds to A with $L-1$ rows, hence $\det(A_{LL}) = d_{L-1}$. Summing up, by using the convention $d_0 = 1$,

$$(33) \quad a_{iL} = d_L^{-1}(1-\beta)^{L-i}d_{i-1}, \quad i = 1, \dots, L$$

As for a_{i1} , we can exploit a certain symmetry of A : using the notation $A = A_\beta$, $[A_\beta]^T = A_{1-\beta}$, so that $A_\beta^{-1} = (A_{1-\beta}^{-1})^T$ to find that

$$(34) \quad a_{i1} = d_L^{-1}\beta^{i-1}d_{L-i}, \quad i = 1, \dots, L.$$

Plugging in (33) and (34) into (32), one obtains

$$\theta_i = \frac{\beta^i d_{L-i}}{d_L} \lambda + \frac{(1-\beta)^{L-i+1} d_{i-1}}{d_L} \mu$$

and the thesis follows by using (30) and (31). By direct calculation, one can check that θ_i is a convex linear combination of λ and μ . Alternatively, one can use directly equation (32). Let $\mathbf{1} = e_1 + \dots + e_L = (1, \dots, 1)$. Since $A\mathbf{1} = (\beta, 0, \dots, 0, (1-\beta))$, we have $\beta A^{-1}e_1 + (1-\beta)A^{-1}e_L = \mathbf{1}$, that is $\beta a_{i1} + (1-\beta)a_{iL} = 1$ for any $1 \leq i \leq L$. Finally, $\beta a_{i1}, (1-\beta)a_{iL} \geq 0$ as a simple calculation reveals. This completes the proof.

As for (27), the result for $\beta = 1/2$ is straightforward. When $\beta \neq 1/2$, it is convenient to write

$$\theta_i^{(L)} = \frac{(1-\beta)^{L+1} \left(\frac{\beta}{1-\beta}\right)^i - \beta^{L+1}}{(1-\beta)^{L+1} - \beta^{L+1}} \lambda + \frac{(1-\beta)^{L+1}}{(1-\beta)^{L+1} - \beta^{L+1}} \left[1 - \left(\frac{\beta}{1-\beta}\right)^i \right] \mu$$

We have

$$\sum_{i=1}^L \theta_i^{(L)} = \frac{1}{(1-\beta)^{L+1} - \beta^{L+1}} \left\{ \left[(1-\beta)^{L+1} \sum_{i=1}^L \left(\frac{\beta}{1-\beta}\right)^i - L\beta^{L+1} \right] \lambda \right.$$

$$\begin{aligned}
& + \left[L(1-\beta)^{L+1} - (1-\beta)^{L+1} \sum_{i=1}^L \left(\frac{\beta}{1-\beta} \right)^i \right] \mu \Big\} \\
& = \frac{(1-\beta)^{L+1} \mu - \beta^{L+1} \lambda}{(1-\beta)^{L+1} - \beta^{L+1}} L + \frac{(1-\beta)^{L+1}}{(1-\beta)^{L+1} - \beta^{L+1}} \sum_{i=1}^L \left(\frac{\beta}{1-\beta} \right)^i (\lambda - \mu)
\end{aligned}$$

The result follows by the formula of the sum of the first L terms of a geometric series,

$$\sum_{i=1}^L \left(\frac{\beta}{1-\beta} \right)^i = \frac{\beta}{1-2\beta} \frac{(1-\beta)^L - \beta^L}{(1-\beta)^L}.$$

The simplification (11) can be derived from (26) by using $x^2 - y^2 = (x-y)(x+y)$. \square

Lemma A.2. Let $\theta_i^{(L)}$ be defined as in (26) with $\mu = \mu_L$ and $\{\mu_L\}_{L \geq 1}$ a decreasing sequence of positive real numbers such that $\mu_L \rightarrow 0$ as $L \rightarrow \infty$. Then $\theta_i^{(L)} \geq \theta_i^{(L-1)}$ for L large enough and any $i = 1, \dots, L$.

Proof. Consider the case of $\beta \neq 1/2$ and, as a short hand notation, let $p = \beta/(1-\beta)$ and $c = [(1-\beta)^{L+1} - \beta^{L+1}][(1-\beta)^L - \beta^L]$. We have

$$\begin{aligned}
\theta_i^{(L)} - \theta_i^{(L-1)} &= \left\{ \frac{(1-\beta)^{L+1} p^i - \beta^{L+1}}{(1-\beta)^{L+1} - \beta^{L+1}} - \frac{(1-\beta)^L p^i - \beta^L}{(1-\beta)^L - \beta^L} \right\} \lambda \\
&+ \left\{ \frac{(1-\beta)^{L+1}}{(1-\beta)^{L+1} - \beta^{L+1}} \mu_L - \frac{(1-\beta)^L}{(1-\beta)^L - \beta^L} \mu_{L-1} \right\} (1-p^i) \\
&= \frac{(1-\beta)^L \beta^L}{c} \{ -(1-\beta)p^i - \beta + 1 - \beta + \beta p^i \} \lambda \\
&+ \frac{1}{c} \{ (1-\beta)^{2L+1} (\mu_L - \mu_{L-1}) - (1-\beta)^L \beta^L [(1-\beta)\mu_L - \beta\mu_{L-1}] \} (1-p^i) \\
&= \frac{(1-\beta)^L \beta^L}{c} (1-2\beta)(1-p^i) \lambda + \frac{1}{c} \left\{ (1-\beta)^{2L+1} (\mu_L - \mu_{L-1}) \right. \\
&\quad \left. - (1-\beta)^L \beta^L (1-2\beta) \left[\mu_L + \frac{\beta}{1-2\beta} (\mu_L - \mu_{L-1}) \right] \right\} (1-p^i) \\
&= \frac{(1-\beta)^L \beta^L (1-p^i)}{c} \{ (1-2\beta)(\lambda - \mu_L) + [(1-\beta)p^{-L} - \beta] (\mu_L - \mu_{L-1}) \}.
\end{aligned}$$

Note that $c \geq 0$ for any $\beta \neq 1/2$. When $\beta < 1/2$, $1-p^i \geq 0$, $(1-2\beta) \geq 0$ and $\lambda \geq \mu_L$ for L large since $\mu_L \rightarrow 0$. As for the second term $[(1-\beta)p^{-L} - \beta] (\mu_L - \mu_{L-1})$ in curly brackets, $(1-\beta)p^{-L} - \beta \uparrow \infty$ and $\mu_L - \mu_{L-1} \geq 0$ as L gets large when μ_L is decreasing. When $\beta > 1/2$, $1-p^i \leq 0$, $(1-2\beta) \leq 0$ and $\lambda \geq \mu_L$ for L large since $\mu_L \rightarrow 0$. Also $(1-\beta)p^{-L} - \beta \downarrow -\beta$ and $\mu_L - \mu_{L-1} \geq 0$ given the monotonicity of $\{\mu_L\}_{L \geq 1}$. So also in this case $\theta_i^{(L)} - \theta_i^{(L-1)} \geq 0$. When

$\beta = 1/2$,

$$\begin{aligned}\theta_i^{(L)} - \theta_i^{(L-1)} &= \left\{ \frac{L-i+1}{L+1} - \frac{L-i}{L} \right\} \lambda + \left\{ \frac{i}{L+1} \mu_L - \frac{i}{L} \mu_{L-1} \right\} \\ &= \frac{1}{L(L+1)} [i(\lambda - \mu_{L-1}) + iL(\mu_L - \mu_{L-1})] \geq 0\end{aligned}$$

for L large since $\lambda \geq \mu_{L-1}$ and $\mu_L - \mu_{L-1} \geq 0$. \square

Proposition A.3. *Let \tilde{M}_L have transitions (15), and M_L as in Definition (6) with $\mu = \mu_L$. For any $L < \infty$, M_L is embedded in \tilde{M}_L at the Markov times σ_k in the sense that the times $\{\sigma_k, n \geq 1\}$ occur infinitely often in \mathbb{N} and*

$$\mathbb{P}\left(\rho_L\left(\tilde{M}_L(\sigma_k)\right) = m' \middle| \rho_L\left(\tilde{M}_L(\sigma_{k-1})\right) = m\right) = p_L(m'|m).$$

for all $k \in \mathbb{N}$ and $m, m' \in \mathbb{Z}_+^L$.

Proof. Let L_i (resp. R_i) denote the event that the i th transition after σ_{k-1} involves components m_1, \dots, m_L (resp. m_{L+1}, m_{L+2}, \dots). Denote also by $\mathbb{P}_{k-1,m}(\cdot)$ the conditional probability $\mathbb{P}(\cdot | M(\sigma_{k-1}) = m)$ and $\mathbb{E}_{k-1,m}(\cdot)$ for the respective conditional expectation. Given $m \in \mathbb{Z}_+^\infty$, we have

$$\mathbb{P}_{k-1,m}(\sigma_k = \sigma_{k-1} + 1) = \mathbb{P}_{k-1,m}(L_1) = \frac{\eta_L + \sum_{i=1}^L i m_i}{\eta_L + |m| - (1 - \beta)(L + 1)m_{L+1}}$$

where $\eta_L = \beta\lambda + (1 - \beta)\mu_L$ and, for $h \geq 2$,

$$\mathbb{P}_{k-1,m}(\sigma_k = \sigma_{k-1} + h) = \mathbb{P}_{k-1,m}(L_h | R_1, \dots, R_{h-1}) \prod_{\ell=1}^{h-1} \mathbb{P}_{k-1,m}(R_\ell | R_1, \dots, R_{\ell-1}).$$

The denominator of $\mathbb{P}_{k-1,m}(L_h | R_1, \dots, R_{h-1})$ depends on the intermediate transitions and is thus random. We can factorise

$$1 - \sum_{h=1}^H \mathbb{P}_{k-1,m}(\sigma_k = \sigma_{k-1} + h) = \prod_{h=1}^H [1 - \mathbb{P}_{k-1,m}(L_h | R_1, \dots, R_{h-1})]$$

and the set $\{\sigma_n, n \geq 1\}$ has an infinite number of terms as long as

$$\sum_{h \geq 1} \mathbb{P}_{k-1,m}(L_h | R_1, \dots, R_{h-1}) \rightarrow \infty.$$

The latter holds since

$$\mathbb{P}_{k-1,m}(L_h | R_1, \dots, R_{h-1}) \geq \frac{\eta_L + \sum_{i=1}^L i m_i}{\eta_L + |m| + h - 1}.$$

Since now the transition at step h is the first to involve the first L components, whose configuration has not changed from $\rho_L(M(\sigma_{k-1}))$, the numerator of the respective probability can be isolated to write

$$\begin{aligned} & \mathbb{P}_{k-1,m}(\sigma_k = \sigma_{k-1} + h) \\ &= \left(\eta_L + \sum_{i=1}^L im_i \right) \mathbb{E}_{|n,m}(T_h^{-1} | R_1, \dots, R_{h-1}) \prod_{\ell=1}^{h-1} \mathbb{P}_{k-1,m}(R_\ell | R_1, \dots, R_{\ell-1}) \end{aligned}$$

where

$$T_h = \eta_L + \sum_{i \geq 1} i M_i(\sigma_{k-1} + h - 1) - \beta(L+1) M_{L+1}(\sigma_{k-1} + h - 1)$$

denotes the denominator in $\mathbb{P}_{k-1,m}(L_h)$ and $M_i(h)$ the random variable for the i th component of $M(h)$. A similar derivation for the transition $m - e_i + e_{i+1}$, $i = 1, \dots, L$, occurring at step h after σ_{k-1} , leads to writing

$$\begin{aligned} & \mathbb{P}_{k-1,m}(\rho_L(M(\sigma_{k-1} + h)) = \rho_L(m - e_i + e_{i+1}), \sigma_k = \sigma_{k-1} + h) = \\ &= \beta im_i \mathbb{E}_{k-1,m}(T_h^{-1} | R_1, \dots, R_{h-1}) \prod_{\ell=1}^{h-1} \mathbb{P}_{k-1,m}(R_\ell | R_1, \dots, R_{\ell-1}), \end{aligned}$$

from which, in turn,

$$\mathbb{P}(\rho_L(M(\sigma_k)) = \rho_L(m - e_i + e_{i+1}) \mid M(\sigma_{k-1}) = m, \sigma_k = \sigma_{k-1} + h) = \frac{\beta im_i}{\eta + \sum_{i=1}^L im_i}.$$

An analogous statement can be derived with a similar argument for all other transitions involving one of the first L components, which, in view of the independence on h of the right hand side, leads to the result. \square

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